CAPRA COACHING

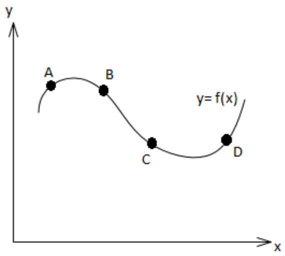
HSC Advanced Maths Exam Booklet:

Differentiation, Geometrical Applications of Calculus

Differentiation, Geometrical Applications of Calculus **Name: ....**

Easy:

1. For the graph of $y = f(x)$ shown, state the point at which $f'(x) < 0$ and $f''(x) > 0$.



A. A
B. B
C. C
D. D

2. The minimum value of $x^2 - 7x + 10$ is:

A. 2
B. $3\frac{1}{2}$
C. $-2\frac{1}{4}$
D. $2\frac{1}{4}$

3. Differentiate with respect to x:

$y = -2(1 + x^3)^4$

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4. Differentiate the following with respect to x.

$(x+1)^2$

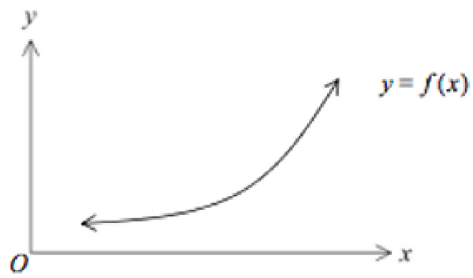
5. Differentiate with respect to x:

$(2x+1)^5$

6. $y = f(x)$ is shown on the number plane.

Which of the following statements is true?

A. $y = f(x)$ is decreasing and concave up.
B. $y = f(x)$ is decreasing and concave down.
C. $y = f(x)$ is increasing and concave up.
D. $y = f(x)$ is increasing and concave down.



7. Differentiate $\frac{x^2}{5x+1}$

8. Consider the function defined by $f(x) = x^3 - 6x^2 + 9x + 2$.

   A. Find $f'(x)$.

   B. Find the coordinates of the two stationary points.

   C. Determine the nature of the stationary points.

D. Sketch the curve $y = f(x)$ for $0 \leq x \leq 4$ clearly labeling the stationary points. (2)

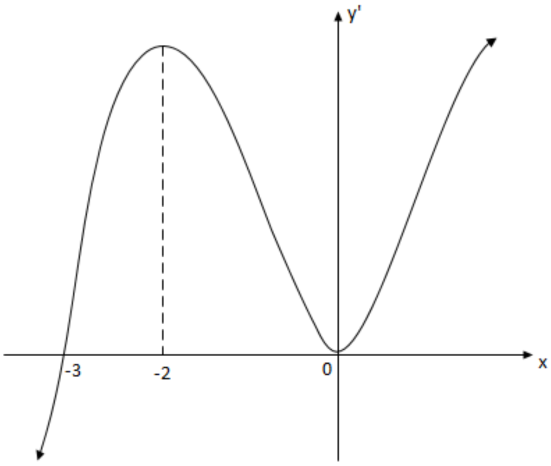
9. Differentiate with respect to x.

A. $\frac{1}{2x^4}$

B. $\frac{\log x}{x}$

C. $cos^2 3x$

10. The graph shows the derivative, y', of a function $y = f(x)$



A. State the values(s) of x where $y = f(x)$ is increasing.

B. State the values(s) of x where $y = f(x)$ has a point of inflexion.

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C. State the values(s) of x where $y = f(x)$ has a minimum turning point.

D. If $f(0) = 1$ sketch the graph of $y = f(x)$.

Medium:

11. A designer T-shirt manufacture finds that the total cost, \$C to make x T-shirts is $C = 4000 + \frac{3x}{20} + \frac{x^2}{1000}$ and the selling price \$E for each T-shirt sold is $30 - \frac{x}{200}$.

    Assuming all but 30 T-shirts are sold:

    A. Show that the total sales, \$S is given by $S = -\frac{x^2}{200} + \frac{603x}{20} - 900$

    B. Show that the profit, \$P is given by $P = -\frac{3}{200}x^2 + 30x - 4900$

C. Find the maximum profit and show that the price for selling each T-shirt must be \$17.50 to achieve this maximum profit.

12. Consider the function $f(x) = x^2\left(\frac{1}{3}x - 1\right)$

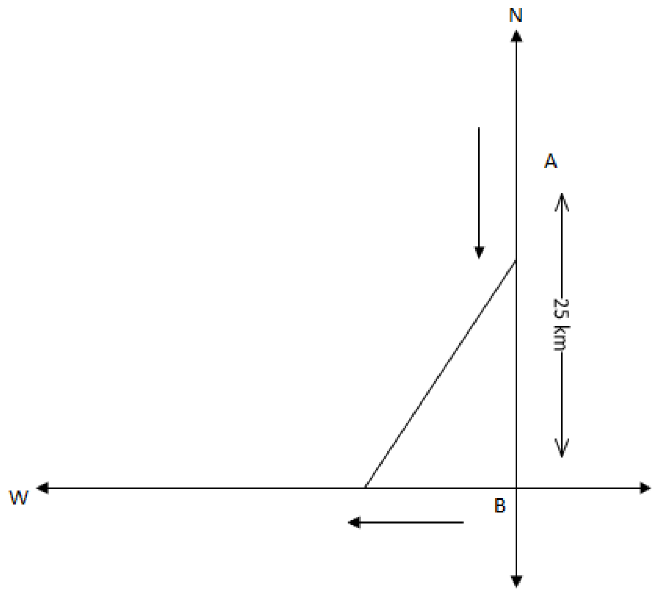
A. Show that $f'^{(x)} = x^2 - 2x$

B. Find any turning points and determine their nature.

C. Find any points of inflexion.

D. Sketch the curve in the domain $-1 \leq x \leq 3$

13. On the compass diagram below, Mary is at position A, 25km due north of position B. John is at B, Mary walks towards B at 4km/h. John moves due west at 6km/h.



A. Show that the distance between Mary and John after t hours is given by: $d^2 = 52t^2 - 200t + 625$

B. Letting $L = d^2$ find the time when L is minimum.

C. Hence find the minimum distance between john and Mary correct to the nearest kilometer.

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14. Consider the curve $y = 2 + 3x - x^2$

A. Find $\frac{dy}{dx}$.

B. Locate any stationary points and determine their nature.

C. For what values of x is the curve concave up?

D. Sketch the curve for $-2 \leq x \leq 2$.

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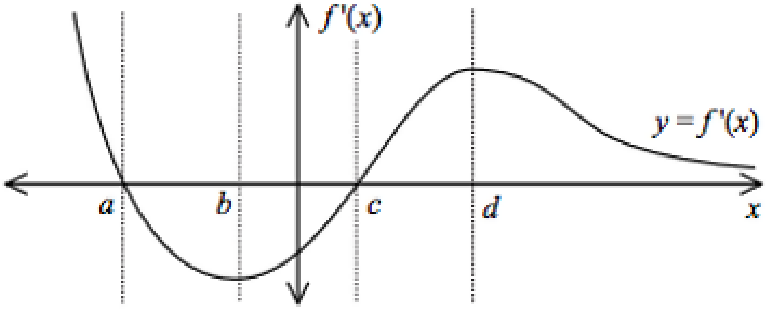
15. The diagram below shows the graph of a gradient function $y = f'(x)$.

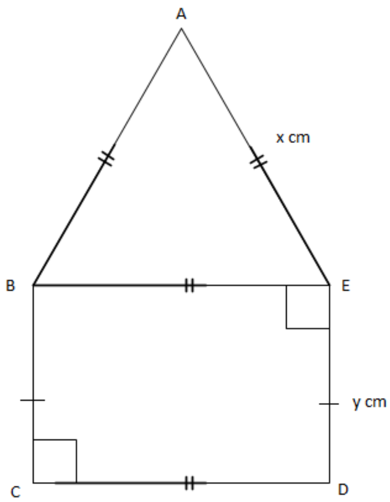


A. Write down the values of x where the curve is stationary.

B. For what value of x will the curve have a maximum turning point?

C. Copy or trace the diagram into your writing booklet. Draw a possible curve for $= f(x)$, clearly showing what is happening to the curve as the values of x increase indefinitely.

16. $ABCDE$ is a pentagon with perimeter 30 cm. The pentagon is constructed with an equilateral triangle ΔABE joining a rectangle $BCDE$.



A. Show that $y = \frac{30-3x}{2}$.

B. Show that the area of ΔABE is $y = \frac{\sqrt{3}x^2}{4}\ cm^2$

C. Hence show that the area of the pentagon is $15x + \frac{(\sqrt{3}-6)x^2}{4}\ cm^2$.

D. Find the exact value of x for which the area of the pentagon will be a maximum. Justify your solution.

17. A rectangular sheet of metal measures 200cm by 100cm. Four equal squares with side lengths x cm are cut out of all the corners and then the sides of the sheet are turned up to form an open rectangular box.

A. Draw a diagram representing this information.

B. Show that the volume of the box can be represented by the equation $V = 4x^3 - 600x^2 + 20\,000x$

C. Find the value of x such that the volume of the tool box is a maximum. (Answer to the nearest cm)

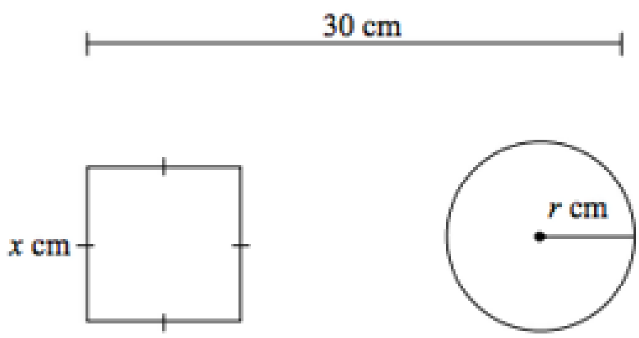
18. Consider the curve $y = x^3 + 3x^2 - 9x - 2$

A. Find any stationary points and determine their nature.

B. Find the coordinates of any point(s) of inflexion.

C. Sketch the curve labelling the stationary points, point of inflexion and y - intercept.

19. A 30 cm length of wire is used to make two frames. The wire is to be cut into two parts. One part is bent into a square of side x Cm and the remaining length is bent into a circle of radius r cm.

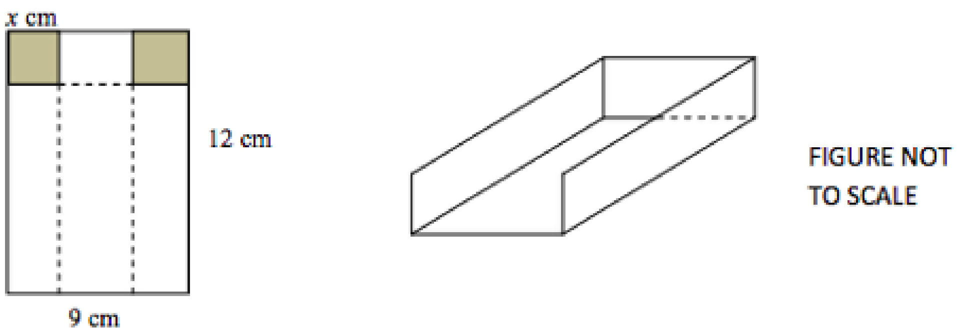


A. The circumference of a circle, C, is found using the formula $C = 2\pi r$Show that the expression for r in terms of x is $r = \frac{15-2x}{\pi}$

B. Show that the combined area, A, of the two shapes can be written as $A = \frac{(4+\pi)x^2 - 60x + 225}{\pi}$

C. Find the value of x for which the combined area of the two frames be minimised. Give your answer correct to 2 significant figures.

20. A rectangular sheet of cardboard measures 12cm by 9cm. From two corners, squares of x cm are removed as Shown. is folded along the dotted to form a tray as shown.



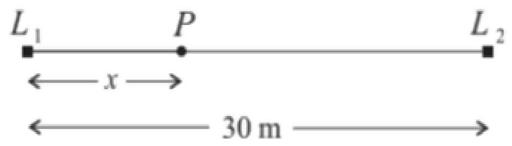
A. Show that the volume, Vcm^3, of the tray is given by $V = 2x^3 - 33x^2 + 108x$.

B. Find the maximum volume of tray

21. Sketch the curve that has the following properties.

A. $f(2) = 1$
B. $f'(2) = 0$
C. $f''(2) = 0$
D. $f'^{(x)} \geq 0$ for all real x.

22. The intensity I produced by a light of power W at a distance x metres from the light is given by $= \frac{W}{x^2}$. Two lights L_1 and L_2, of power W and $2W$ respectively, are positioned 30 metres apart.



A. Write down an expression for the combined intensity I_c of L_1 and L_2 at a point P which is x metres from L_1, as shown in the diagram.

B. Find $\frac{dI_e}{dx}$.

C. Find the distance PL_1, correct to the nearest centimetre, so that combined intensity of L_1 and L_2 is at its minimum.

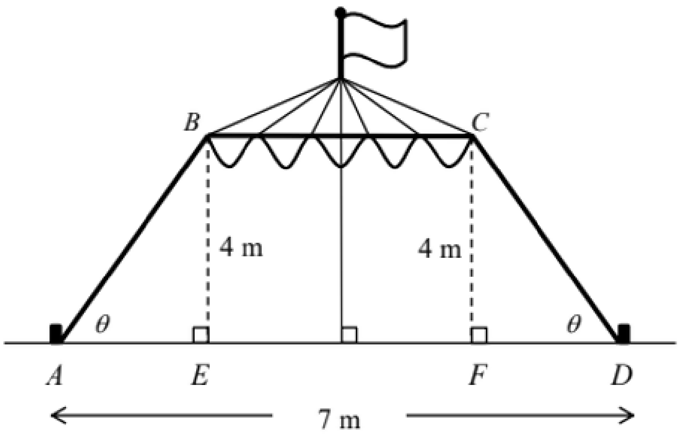
23. The velocity of a train increases from 0 to V at a constant rate a. velocity then remains constant at V for a certain time. After this time the velocity decreases to O at a rate b. Given that the total distance travelled by train is s and the time for the Journey is T.

A. Draw a velocity-time graph for the above information,

B. Show that the time (T) for the journey is given by $T = \frac{s}{V} + \frac{1}{2}V\left(\frac{1}{a} + \frac{1}{b}\right)$.

C. When a, b and s are find the speed that will minimize the time for the journey.

24. A circus marquee is supported by a center pole, and secured by the rope $ABCD$. George, a curious monkey, wants to know how quickly he can climb up and down the marquee along the rope $ABCD$.



A. If $BE = CF = 4$ metres and $AD = 7$ meter, show that $BC = 7 - \frac{8}{\tan\theta}$

B. Hence show that the total time (t) needed for George to climb up and down the marquee, in minutes, is given by: $t = \frac{7}{4} + \frac{4-2\cos\theta}{\sin\theta}$

C. Given that $0 < \theta < \frac{\pi}{2}$, find value of θ for which the time taken by George to climb up and down the marquee is a minimum.

25. An open cylindrical water tank has base radius x meters and height h meters. Each square meter of base cost a dollars to manufacture and each square metre of the curved surface costs b dollars, where a and b are constants. The combined cost of the base and curved surface is c dollars.

A. Find c in the term of a, b, x and h (Note that the curved surface has area $2\pi xh$.)

B. Show that the volume V of the tank in cubic metres is given by $V = \frac{x}{2b}(c - \pi a x^2)$.

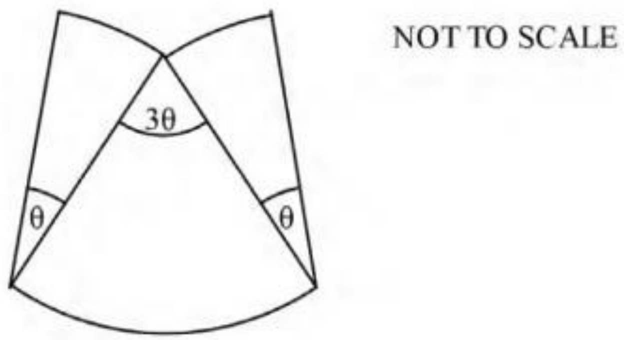
C. If x can vary, prove that V is maximised when the cost of the base is $\frac{c}{3}$ dollars.

26. Let f and g be functions where $f'(2) = 2$, $g(2) = 1$, $f'(1) = 3$ and $g'(2) = -2$.

What is the gradient of the tangent to the curve $y = f[g(x)]$ at the point where $x = 2$?

A. 6
B. 3
C. 2
D. -6

27. The diagram below shows a design to be used on a new brand of jam. The design consists of three circular sectors each of radius r cm. The angle of two of the sectors is θ radian and the angle of the third sector is 3θ radian as shown.



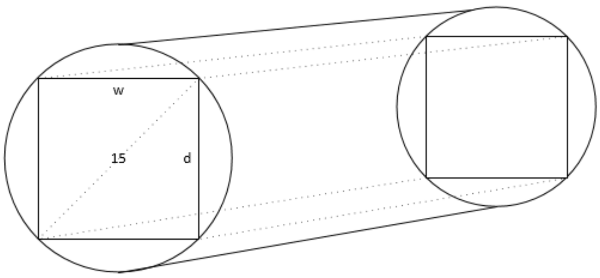
Given that the area of the design is 25 cm^2,

A. Show that $= \frac{10}{r^2}$.

B. Find the external perimeter of the design, P, in terms of r.

C. Given that r can vary, find the value of r for which P is minimum.

28. A rectangular beam of width w cm and depth d cm is cut from a cylindrical pine log as shown.
The diameter of the cross-section of the log (and hence the diagonal of the cross-section of the beam is 15 cm). The strength S of the beam is propotional to the product of its width and the square of its depth, so that $S = kd^2w$.



A. Show that $S = k\,(225\,w - w^3)$.

B. Find numerically the dimensions that will maximize the strength of the beam. Justify your answer.

C. Find the strength S of the beam if its cross sectional area is a square with diagonal 15 cm.

D. Express as a percentage, how much stronger will beam of strength be in comparison to the square beam in part iii to the nearest %.

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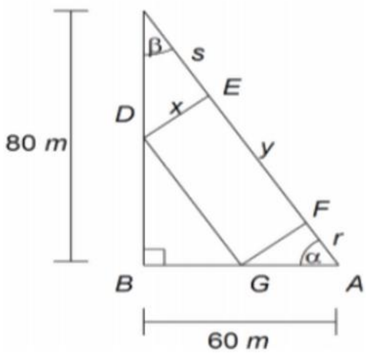
29. A lot of land has the form of a right triangle, with perpendicular sides 60 and 80 meters long.



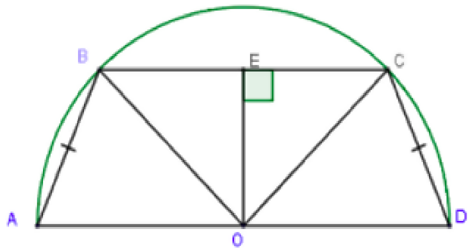
A. Show that $r = \frac{3}{4}\, x \text{ and } s = \frac{4}{3}\, x$

B. Show that $y = 100 - \frac{25}{12}\, x$

C. Find the length and width of the largest rectangular building that can be erected facing the hypotenuse of the triangle.

Hard:

30. An isosceles trapezium ABCD is drawn with its vertices on a semi-circle center O and diameter 20 cm (see diagram). OE is the altitude of ABCD.



A. Prove that $\Delta BOE = \Delta\, COE$

B. Hence of otherwise, show that the area of the trapezium ABCD is given by: $A = \frac{1}{4}\,(x+20)\sqrt{400-x^2}$ Where x is the length of BC

C. Hence find the length of BC so that the area of the trapezium ABCD is a maximum.

31. Evaluate: $\lim_{x \to 16} \frac{x-16}{\sqrt{x}-4}$

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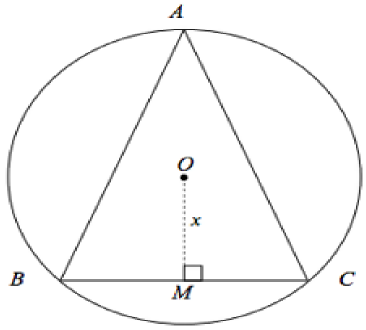
32. A function is defined by the following properties: $y = 0$ when $x = 1$; $\frac{dy}{dx} = 0$ when $x = -3$, 1 and 5; and $\frac{d^2y}{dx^2} > 0$ for $x \leq -1$ and $1 < x < 3$.

Sketch a possible graph of the function.

33. An isosceles triangle ABC with $AB = AC$ is inscribed in a circle centre O and of radius R units.

Given that $OM = x$ units, $OM \perp BC$ and M is the midpoint of BC,



A. Show that the area of ΔABC, S square units, is given by: $S = (R + x)\sqrt{R^2 - x^2}$

B. Hence show that the triangle with maximum area is an equilateral triangle.

34. When a ship is travelling at a speed of $v\ km/hr$, its rate of consumption of fuel in tonnes per hours given by $125 + 0.004v^3$.

A. Show that on a voyage of 5000km at a speed Of $v\ km/hr$ the formula for the total fuel used, T tonnes, is given by: $T = \frac{625000}{v} + 20v^2$

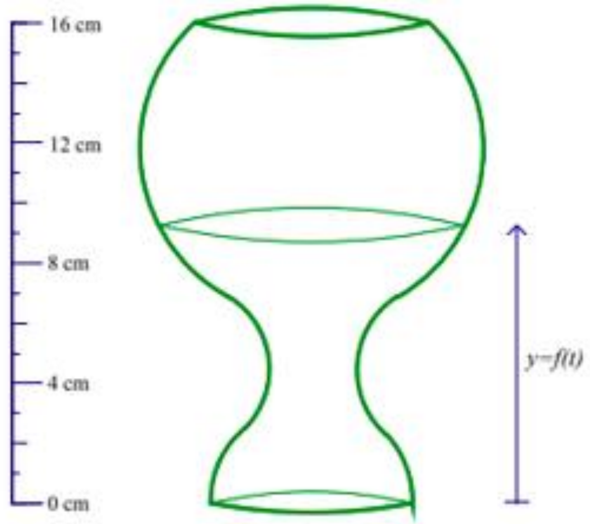
B. Hence find the speed for the greatest fuel economy and the amount of fuel used at this speed. (Justify your answer.)

35.

A. Justify the graph of $f(x) = x - \frac{1}{x^2}$ is always concave down.

B. Sketch the graph $(x) = x - \frac{1}{x^2}$, showing all intercept(s) and stationary point(s).

36. The diagram shows a 16 cm high Wine glass that is being filled with water at a constant rate (by volume). Let $y = f(t)$ be the depth of the liquid in the glass as a function of time.



A. Write down the approximate depth y_1, at which $\frac{dy}{dt}$ is a minimum.

B. Write down the approximate depth y_2, at which $\frac{dy}{dt}$ is a maximum.

C. If the glass takes 8 seconds to fill, graph $y = f(t)$ and identify any points on your graph where the concavity changes.

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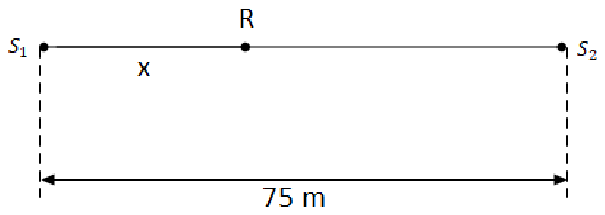
37. Find $\lim_{h \to 0} \left(\frac{4^h - 1}{2^h - 1}\right)$

38.

A. Show that $\frac{ds_\infty}{dx} = \frac{-x^2+2x+1}{(1-x)^2}$

B. Find the minimum value of the sum to infinity. Justify your answer.

39. At a music concert two speaker towers are placed 75 metres apart. The intensity of sound produced by a speaker tower of power P at a distance x metres from the tower is given by $I = \frac{4P}{\pi r^2}$.



The speakers in S_1 , have a power output of P but the older speakers in tower S_2 produce 25% less power. Rebecca R stands in between the two towers and x metres from tower S_1 as shown in the diagram above.

A. Show that the sound intensity produced by speaker tower S_2 at point R is $I = \frac{3P}{\pi(75-x)^2}$.

Given that the total sound intensity from both speaker towers I_T at point R is

$I = \frac{4P}{\pi r^2} + \frac{3P}{\pi(75-x)^2}$,

B. Find $\frac{dI_T}{dx}$.

C. Hence find the value of x which minimizes the sound intensity So that Rebecca knows where best to stand to enjoy the concert. Give your answer to three significant figures.

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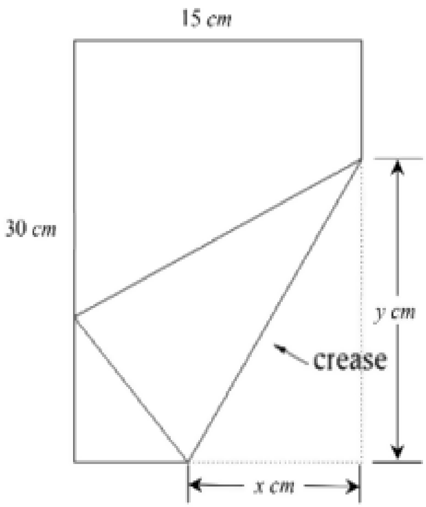
40. The derivative of $y = x^x$ is

A. $x \cdot x^{x-1}$
B. $(1 + \log_e x) \cdot x^x$
C. $2x \cdot x^x$
D. $(x \log_e x) \cdot x^x$

41. A rectangular piece of paper is 30 centimetres high and 15 centimetres wide. The lower right-hand corner is folded over so as to reach the leftmost edge of the paper. Let x be the horizontal distance folded and y be the vertical distance folded as shown in the diagram.



A. By considering the areas of three triangles and trapezium that make up the total area of the paper, show

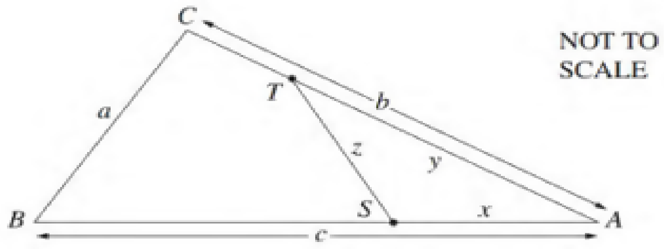
$$y = \frac{x\sqrt{15(2x-15)}}{2x-15}$$

B. Show that the crease, C. is found by the expression

$$C = \sqrt{\frac{2x^3}{2x - 15}}$$

C. Hence, find the minimum length of *C.*

42. The diagram shows a triangular piece of land ABC with dimensions $AB = c$ metres, $AC = b$ metres and $BC = a$ metres where $a \leq b \leq c$.
The owner of the land wants to build a straight fence to divide the land into two pieces of equal area. Let S and T be the points On AB and AC respectively so that ST divides the land into two equal areas. Let $AS = x$ metres, $AT = y$ metres and $ST = z$ metres.



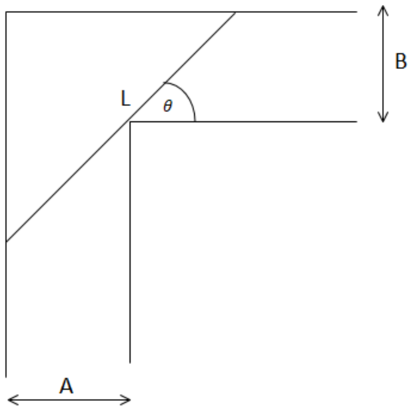
A. Show that $xy = \frac{1}{2}bc$.

B. Use the cosine rule in triangle AST to show that $z^2 = x^2 + \frac{b^2c^2}{4x^2} - bc\cos A$.

C. Show that the value of z^2 in the equation in part (ii) is a minimum when $x = \sqrt{\frac{bc}{2}}$.

D. Show that the minimum length of the fence is $\sqrt{\frac{(p-2b)(p-2c)}{2}}$ metres, where $P = a + b + c$. (You may assume that the value of x given in part (iii) is feasible).

43. Two corridors meet at right angles. One has a width of A metres, and the other has a width of B metres. Mario the plumber wants to find the length of longest pipe, L that he can carry horizontally around the corner as seen in the diagram below. Assume that the pipe has negligible diameter.



A. Show that $L = A\sec\theta + B\ cosec\ \theta$.

B. Explain why the length of the pipe, L, needs to be minimized in order to obtain the length of longest pipe that can be carried around the corner.

C. Hence show that When $\tan\theta = \sqrt[3]{\frac{B}{A}}$ the solution will be minimized. (You do **NOT** need to test to show that the solution will give a minimum length).

D. Hence using the result in part (a), show that the length of the largest pipe that can be carried around the corner is $L = \left(A^{\frac{2}{3}} + B^{\frac{2}{3}}\right)^{\frac{2}{2}}$

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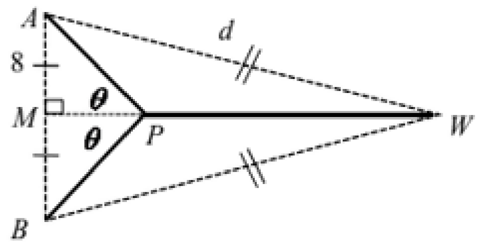
44. The diagram below shows two towns A and B that are 16 km apart, and each at a distance of d km from a water well at W.
Let M be the midpoint of AB, P be a point on the line segment MW, and $\theta = \angle APM = \angle BPM$.
The two towns are to be supplied with water from W, via three straight water pipes: PW, PA and PB as shown below.

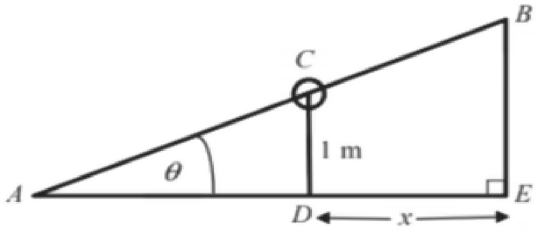


A. Show that the total length of the water pipe L is given by$L = 8f(\theta) + \sqrt{d^2 - 64}$
Where, $f(\theta)$ is given in part (b) above.

B. Find the minimum value L if $d = 20$.

C. If $d = 9$, show that the minimum value of L cannot be found by using the same methods as used in part (ii).
Explain briefly how to find the minimum value of L in this case.

45. The diagram below shows a straight rod AB of length 8 m hinged to the ground at A.
CD is a rod of 1m.
The end C is free to slide along AB while the end is touching the ground floor such that CD is perpendicular to the ground.



Let $DE = x\ m$ and $\angle BAE = \theta$, where $\frac{1}{8} < \sin\theta < 1$.

A. Express x in terms of θ .

B. Find the maximum value of x as θ varies.

C. Let M be the area of trapezium $CDEB$. Show that
$M = \left(\frac{1 + 8\sin\theta}{2}\right)(8\cos\theta - \cot\theta)$

D. Does M attain a maximum when x reaches its maximum? Justify your answer.

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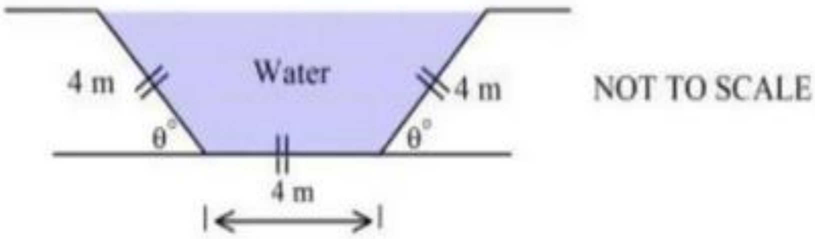
46. An irrigation channel has a cross-section in the shape of a trapezium as shown in the diagram. The bottom and sides of the trapezium are 4 metres long.
Suppose that the sides of the channel make an angle of θ with the horizontal where $\theta \leq \frac{\pi}{2}$.



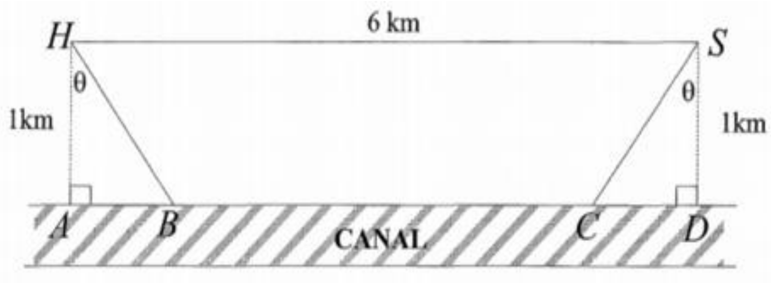
A. Show that the cross-sectional area is given by $A = 16(\sin\theta + \cos\theta - 1)$.

B. Show that $\frac{dA}{d\theta} = 16(2\cos^2\theta + \cos\theta - 1)$

C. Hence, show that the maximum cross-sectional area occurs when $\theta = \frac{\pi}{3}$.

D. Hence, find the maximum area of the irrigation channel correct to the nearest square metre.

47.

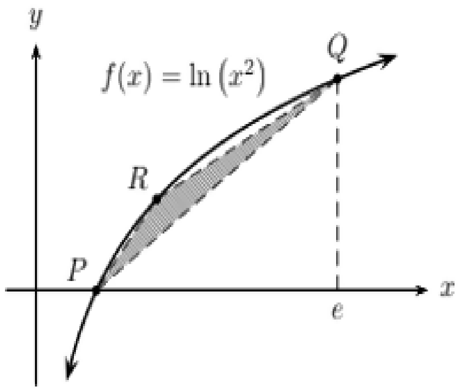


The diagram above that the distance between a boy's home H and his school S is *6 km*. A canal $ABCD$ is *1km* from both his home and winter canal is frozen, so he take an alternate route $HBCS$, walking HB, skating BC and walking CS. His walking is *4km/h* and his skating is *12km/h*. Let $\angle AHB = \angle DSC = \theta$.

A. Show that the time taken for this alternate route is $T = \frac{1}{2\cos\theta} + \frac{1}{2} - \frac{\tan\theta}{6}$.

B. Find, to the nearest minute, the value of θ which minimizes the time taken for the journey to school.

48. The diagram shows the graph of the function $y = \ln(x^2), (x > 0)$.
The points $P(1,0), Q(e,2)$ and $R(t, \ln t^2)$ all lie on the curve.
The area of ΔPQR is maximum when the tangent at R is parallel to the line through P and Q.



A. Find gradient of the line through P and Q.

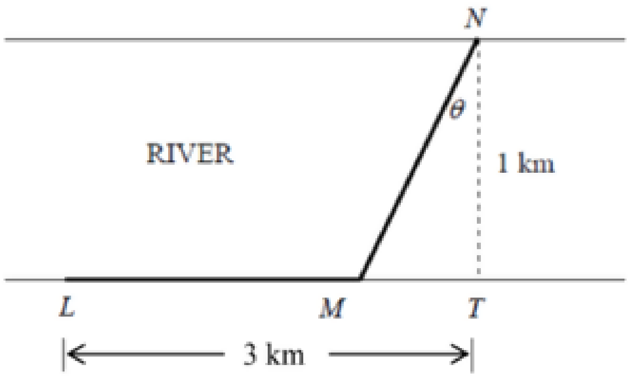
B. Find the value of t that gives the maximum area for ΔPQR.

C. Hence find the maximum area of ΔPQR

49. If the straight line $y = mx$ is a tangent to the curve $= e^{\frac{x}{2}}$, find the exact value of m. Clearly show your working.

Differentiation, Geometrical Applications of Calculus **Name: ……………………….**

50. A cable link is to be constructed between two points L and N, which are situated on opposite banks of a river of width $I\ km$. L lies 3 km upstream from N. It costs three times as much to lay a length of cable underwater as it does to lay the same length overland. The following diagram is a sketch of the cables where θ is the angle NM makes with the direct route across the river.



Not to scale

A. Show $MN = \sec\theta$ and $MR = \tan\theta$.

B. If segment LM costs c dollars per km, show the total cost *(T)* of laying the cable from L to M to N is given by: $T = 3c - c\tan\theta + 3c\sec\theta$

C. At what angle, θ, should the cable cross the river in order to minimise the total cost of laying the cable?